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离散 ϕ -Laplace 问题的正解*

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摘　要：考虑如下离散的 ϕ -Laplace 边值问题 $\begin{cases}\Delta(\phi(\Delta u(k-1)))+\lambda p(k)f(u(k))=0, k\in\{1,2,\cdots,T\},\\ u(0)=u(T+1)=0\end{cases}$ 其中，$T>5$ 是给定的正整数，λ 是正参数，ϕ 是从 $\mathbf{R}$ 到 $\mathbf{R}$ 上的单调递增的奇同胚映射。在较弱的条件 $\liminf\limits_{u\to\infty}\dfrac{f(u)}{\phi(u)}\in(0,\infty]$ 下，利用锥上的不动点定理，建立了问题至少存在一个正解的结果，并给出了参数 λ 的显式开区间。

关键词：离散 ϕ -Laplace 问题；正解；半正问题

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Positive solutions of discrete ϕ -Laplacian problems

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Abstract: The discrete ϕ -Laplacian boundary problem $\begin{cases}\Delta(\phi(\Delta u(k-1)))+\lambda p(k)f(u(k))=0,\\ \quad k\in\{1,2,\cdots,T\},\\ u(0)=u(T+1)=0\end{cases}$ is studied, where $T>5$ is a given positive integer, λ is a nonnegative parameter and ϕ is an odd and increasing homeomorphism from $\mathbf{R}$ onto $\mathbf{R}$. Applying the fixed point theorem in cones, it is proved under the weakened condition $\liminf\limits_{u\to\infty}\dfrac{f(u)}{\phi(u)}\in(0,\infty]$ that the problem has at least one positive solution for λ belonging to an explicit open interval.

Key words: discrete ϕ -Laplacian problem; positive solution; semipositone problem

For $a,b\in Z$ with $a<b$, let $[a,b]_Z=\{a,a+1,a+2,\cdots,b-1,b\}$. Let $\phi:\mathbf{R}\to\mathbf{R}$ be an odd and increasing homeomorphism. For $u:[a,b]_Z\to\mathbf{R}$, we define

$$L[u](\cdot):=\Delta(\phi(\Delta u(\cdot)))$$

where Δ denotes forward difference operator, i. e., $\Delta u(k)=u(k+1)-u(k), k\in[a,b-1]_Z$. We consider the positive solutions of the following discrete ϕ -Laplacian boundary value problem

$$\begin{cases}L[u](k-1)+\lambda p(k)f(u(k))=0, k\in[1,T]_Z,\\ u(0)=u(T+1)=0\end{cases}\tag{1}$$

where $T>5$ is a given positive integer, λ is a nonnegative parameter and $p(k)>0$ on $[1,T]_Z$.

Recently, the positive solutions of ϕ -Laplacian boundary value problems (i. e., generalized p-Lapla-

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cian boundary value problems) have been widely studied. For differential ϕ-Laplacian problems, we refer reader to [1 − 8] for some references. For the discrete case, there are less study results than continuous case [9-10]. We also refer to Cabada[11] and Cabada and Espinar[12] for the results of upper and lower solutions method, and Bondar[13] for the existence and uniqueness results by the fixed point theory of contraction mapping. In [9 − 10], the existence and multiplicity of positive solutions of discrete ϕ-Laplacian boundary value problems were studied and f was assumed to be nonnegative. In this paper, we consider the semipositone case (i. e. the case $f(0) < 0$). Our interest is the explicit open intervals of λ such that (1) has at least one positive solution.

Semipositone problems arise in many different areas of applied mathematics and physics, see e. g. [14 − 16]. The study methods, in addition to fixed point method, more is the bifurcation theory, shooting method and time mapping analysis method. See [16 − 19] and the references therein. For the results of applying fixed point method, we refer [2, 19 − 27] for some references. The study of this paper is motivated by Hai, Schmitt and Shivaji[2], in which the existence of positive solutions of differential ϕ-problems was studied by fixed point method with f satisfying ϕ-superlinear condition (i. e. $\lim\limits_{u\to\infty}\dfrac{f(u)}{\phi(u)} = \infty$) and ϕ satisfying

(C1) ϕ^{-1} is concave on $\mathbf{R}^+$, and for each $\delta > 0$, there exists $A_\delta > 0$ such that $\phi^{-1}(\delta u) \geqslant A_\delta \phi^{-1}(u)$, $u \in \mathbf{R}^+$ and $\lim_{\delta\to\infty} A_\delta = \infty$.

In our discussions to problem (1), the concavity of ϕ^{-1} is reserved and the following condition is needed.

(C2) There exist increasing homeomorphisms ψ, $\varphi : (0,\infty) \to (0,\infty)$, such that for all $\mu > 1$,

$$\psi(\mu) \leqslant \frac{\phi(\mu x) - \phi(\mu y)}{\phi(x) - \phi(y)} \leqslant \varphi(\mu),$$
$$\forall x, y \in \mathbf{R} : x \neq y$$

Note that condition (C2) implies

$$\psi(\mu)\phi(x) \leqslant \phi(\mu x) \leqslant \varphi(\mu)\phi(x)$$
$$\text{for all } x > 0 \text{ and } \mu > 1 \qquad (2)$$

which was used in [3, 7] for all $x > 0$, $\mu > 0$. By using the inequality (2), the ϕ-superlinear condition imposed on f in [2] can be weakened by a more general condition (see (A2) below). Now, we state our assumptions as follows.

(A1) $f : \mathbf{R} \to \mathbf{R}$ is continuous and there exists $M > 0$ such that $f(u) \geqslant -M$ holds for all $u \geqslant 0$;

(A2) $\liminf_{u\to\infty} \dfrac{f(u)}{\phi(u)} = f^\infty \in (0,\infty]$;

(A3) ϕ^{-1} is concave on $\mathbf{R}^+$.

The following fixed point theorem in cones will be used to prove our main results.

Lemma 1[28] Let E be a Banach space and let K be a cone in E such that $\|\cdot\|$ is monotone with respect to the cone K. Let $F : K \to K$ be a completely continuous operator. Assume there exist positive constants r, R, and $\rho \in K$, $\beta \in K$ with $0 < r < R$, $\|\rho\| < r$, $\|\beta\| > R$ such that

(i) For each $0 < \theta < 1$, all solutions $u \in K$ of $u = \theta F u + (1 - \theta)\rho$ satisfy $\|u\| \neq r$.

(ii) For each $0 < \theta < 1$, all solutions $u \in K$ of $u = Fu + \theta\beta$ satisfy $\|u\| \neq R$.

Then F has at least one fixed point $u \in K$ with $r \leqslant \|u\| \leqslant R$.

Throughout this paper, we take $E = \{u : [0, T+1]_Z \to \mathrm{R}\}$ equipped with norm $\|u\| = \max\limits_{k\in[1,T]_Z} |u(k)|$.

Denote

$$\underline{p} := \min_{k\in[1,T]_Z} p(k)$$

and

$$\bar{p} := \max_{k\in[1,T]_Z} p(k)$$

We note that a function u of integer variable is said to be concave if $\Delta^2 u(k-1) \leqslant 0$. This is the same as saying that the first difference $\Delta u(k-1)$ is non-increasing. If $\Delta^2 u(k-1) < 0$, then u is said to be strictly concave. For a sum $\sum\limits_{k=s}^{t} u(t)$, if $s > t$, then we say $\sum\limits_{k=s}^{t} u(t) = 0$.

1 Preliminaries

Lemma 2 Let $a, b \in Z$ with $a < b$. Assume $q : [a-1, b+1]_Z \to \mathbf{R}$ and $h : [a, b]_Z \to (0, \infty)$. If u and v satisfy

$$\begin{cases} L[u-q](k-1) - L[v-q](k-1) + h(k) = 0, \\ \qquad k \in [a,b]_Z, \\ (u-v)(a-1) = (u-v)(b+1) = 0 \end{cases} \qquad (3)$$

then $u > v$ on $[a, b]_Z$.

Proof Suppose that the conclusion is not true, then there exists $k^* \in [a,b]_Z$ such that $(u-v)(k^*) = \min\limits_{k\in[a,b]_Z} (u-v)(k) \leqslant 0$. Summing both sides of the equation in (3) from s to k^*, we have

$$\phi(\Delta(u-q)(s-1)) - \phi(\Delta(v-q)(s-1)) = \sum_{k=s}^{k^*} h(k) + \phi[\Delta(u-q)(k^*)] - \phi[\Delta(v-q)(k^*)] > 0$$

for $s \in [a, k^*]_Z$ since $\Delta(u-v)(k^*) \geqslant 0$. Thus,

$\Delta(u-v)(s-1)>0, s\in[a,k^*]_Z$. It follows by summing from a to k^* that $(u-v)(k^*)>0$, a contradiction.

Lemma 3 If w satisfies

$$\begin{cases} L[u](k-1)+\lambda Mp(k)=0, k\in[1,T]_Z,\\ u(0)=u(T+1)=0 \end{cases}\qquad (4)$$

Then $w>0$ on $[1,T]_Z$ and $|\Delta w(k-1)|\leqslant \phi^{-1}(\lambda MT\bar{p})$ on $[1,T+1]_Z$.

Proof By Lemma 2, we know $w>0$ on $[1,T]_Z$. It is easy to see that the unique solution of (4) can be written as

$$w(k)=\sum_{s=1}^{k}\phi^{-1}(C+\sum_{l=s}^{T}\lambda Mp(l)),\quad k\in[0,T+1]_Z$$

where C satisfies $w(T+1)=0$. Then $-\sum_{k=1}^{T}\lambda Mp(k)<C<0$. In fact, if $C\geqslant 0$, then we have $0=w(T+1)\geqslant\sum_{s=1}^{T+1}\phi^{-1}(\sum_{l=s}^{T}\lambda Mp(l))>0$, a contradiction. The another side can be verified similarly. Since $\phi(\Delta w(k-1))=C+\sum_{l=k}^{T}\lambda Mp(l), k\in[1,T+1]_Z$, we have that for $k\in[1,T+1]_Z$,

$$\phi(\Delta w(k-1))<\sum_{l=k}^{T}\lambda Mp(l)\leqslant \sum_{l=1}^{T}\lambda Mp(l)\leqslant\lambda MT\bar{p}$$

and

$$\phi(\Delta w(k-1))>-\sum_{l=1}^{k-1}\lambda Mp(l)\geqslant -\sum_{l=1}^{T}\lambda Mp(l)\geqslant-\lambda MT\bar{p}$$

Lemma 4[2] Let (A3) hold. The following inequalities hold.

$$\phi^{-1}(x+y)\leqslant\phi^{-1}(x)+\phi^{-1}(y)+\phi^{-1}(\mu)\quad \text{for } x\geqslant-\mu, y\geqslant 0, \mu\geqslant 0,$$

$$\phi^{-1}(x-y)\geqslant\phi^{-1}(x)-2\phi^{-1}(y)\quad \text{for } x\geqslant 0, y\geqslant 0.$$

Let $g(u):=f(u)+M$ for $u\geqslant 0$. Define $\tilde{g}(u)=g(u)$ if $u\geqslant 0$ and $\tilde{g}(u)=g(0)$ for $u\leqslant 0$.

Lemma 5 Let (A1) hold. u is a positive solution of (1) if and only if $\tilde{u}=u+w$ is a solution of the following problem

$$\begin{cases} L[\tilde{u}-w](k-1)+L[w](k-1)= \\ -\lambda p(k)\tilde{g}(\tilde{u}(k)-w(k)), k\in[1,T]_Z, \\ \tilde{u}(0)=0=\tilde{u}(T+1) \end{cases}\qquad (5)$$

with $\tilde{u}>w$ on $[1,T]_Z$.

It is easy to see that problem (5) is equivalent to

$$\tilde{u}(k)=\sum_{s=1}^{k}\phi^{-1}\Big(C_{\tilde{u}}-\lambda\sum_{l=1}^{s-1}p(l)\tilde{g}(\tilde{u}(l)-w(l))-\phi(\Delta w(s-1))\Big)+w(k),\quad k\in[0,T+1]_Z\qquad(6)$$

where $C_{\tilde{u}}$ satisfies $\tilde{u}(T+1)=0$. Equivalently,

$$\tilde{u}(k)=\sum_{s=k}^{T}\phi^{-1}\Big(\tilde{C}_{\tilde{u}}-\lambda\sum_{l=s+1}^{T}p(l)\tilde{g}(\tilde{u}(l)-w(l))+\phi(\Delta w(s))\Big)+w(k), k\in[0,T+1]_Z\qquad(7)$$

where $\tilde{C}_{\tilde{u}}$ satisfies $\tilde{u}(0)=0$.

By Lemma 2, $\tilde{u}(k)>0, k\in[1,T]_Z$. Let $\|\tilde{u}\|=\tilde{u}(k_0), k_0\in[1,T]_Z$. Then we have the following inequalities, which is crucial in our proceeding arguments:

$$\lambda\sum_{l=1}^{k_0-1}p(l)\tilde{g}(\tilde{u}(l)-w(l))\leqslant C_{\tilde{u}}\leqslant \lambda\sum_{l=1}^{k_0}p(l)\tilde{g}(\tilde{u}(l)-w(l))\qquad(8)$$

and

$$\lambda\sum_{l=k_0+1}^{T}p(l)\tilde{g}(\tilde{u}(l)-w(l))\leqslant\tilde{C}_{\tilde{u}}\leqslant \lambda\sum_{l=k_0}^{T}p(l)\tilde{g}(\tilde{u}(l)-w(l))\qquad(9)$$

In fact, on the one hand, by

$$0\leqslant\tilde{u}(k_0)-\tilde{u}(k_0-1)=\phi^{-1}\Big(C_{\tilde{u}}-\lambda\sum_{l=1}^{k_0-1}p(l)\tilde{g}(\tilde{u}(l)-w(l))-\phi(\Delta w(k_0-1))\Big)+\Delta w(k_0-1)$$

we know that $C_{\tilde{u}}\geqslant\lambda\sum_{l=1}^{k_0-1}p(l)\tilde{g}(\tilde{u}(l)-w(l))$. On the other hand, by $\Delta\tilde{u}(k_0)\leqslant 0$, we have $C_{\tilde{u}}\leqslant\lambda\sum_{l=1}^{k_0}p(l)\tilde{g}(\tilde{u}(l)-w(l))$. The inequality (9) can be verified similarly.

Let $K=\{u\in E: u(k)\geqslant 0, k\in[0,T+1]_Z\}$, a cone in E. With the help of (6) and (7), define $F:K\to K$ by

$$Fu(k)=\sum_{s=1}^{k}\phi^{-1}\Big(C_u-\lambda\sum_{l=1}^{s-1}p(l)\tilde{g}(u(l)-w(l))-\phi(\Delta w(s-1))\Big)+w(k)=$$

$$\sum_{s=k}^{T}\phi^{-1}\Big(\tilde{C}_u-\lambda\sum_{l=s+1}^{T}p(l)\tilde{g}(u(l)-w(l))+\phi(\Delta w(s))\Big)+w(k)$$

Here, C_u and $\widetilde{C}_u$ satisfy

$$\sum_{s=1}^{T+1}\phi^{-1}\Big(C_u-\lambda\sum_{l=1}^{s-1}p(l)\widetilde{g}(u(l)-w(l))-\phi(\Delta w(s-1))\Big)=0$$

and

$$\sum_{s=0}^{T}\phi^{-1}\Big(\widetilde{C}_u-\lambda\sum_{l=s+1}^{T}p(l)\widetilde{g}(u(l)-w(l))+\phi(\Delta w(s))\Big)=0$$

respectively. Then $F:K\to K$ is continuous and problem (5) has a positive solution if and only if F has a fixed point in K for $\lambda>0$.

2 Main Results

For any given $r>0$, set

$$\lambda_r:=\min\left\{\frac{\phi\Big(\frac{r}{8(T+1)}\Big)}{MT\bar{p}},\frac{\phi\Big(\frac{r}{2(T+1)}\Big)}{h(r)T\bar{p}}\right\} \tag{10}$$

where $h(t)=\sup\limits_{0\leqslant s\leqslant t}\widetilde{g}(s)$. Also, let $\lambda_0:=\dfrac{2\varphi(12)}{\underline{p}f^{\infty}\psi\Big(\frac{T+1}{3}\Big)}$.

If $f^{\infty}=\infty$, then we say $\lambda_0=0$.

Now, we state our main result.

Theorem 1 Let (A1) – (A3) and (C2) hold. Assume that $\lambda_0<\lambda_r$ holds for some $r>0$, then problem (1) has at least one positive solution for $\lambda\in(\lambda_0,\lambda_r)$. Especially, if $f^{\infty}=\infty$, then for each $r>0$ problem (1) has one positive solution for $\lambda\in(0,\lambda_r)$.

Proof We divide the proof into two steps.

Step I: We shall apply Lemma 1 to show that for $\lambda_0<\lambda<\lambda_r$, F has a fixed point $\widetilde{u}$ in K.

First, we show that all solutions $u\in K$ of $u=\theta F(u)$ $(\theta\in(0,1))$ satisfy $\|u\|\neq r$.

Indeed, if there exists $\theta\in(0,1)$ such that a solution u of $u=\theta Fu$ satisfies $\|u\|=r$, then by Lemma 3 and 4 with (8) and (10), we have that for $k\in[0,T+1]_Z$,

$$\begin{aligned}
u(k)&=\theta Fu(k)\leqslant\\
&\theta\sum_{s=1}^{k}\phi^{-1}\Big(\lambda\Big(\sum_{l=1}^{T}-\sum_{l=1}^{s-1}\Big)p(l)\widetilde{g}(u(l)-w(l))-\phi(\Delta w(s-1))\Big)+\theta w(k)=\\
&\theta\sum_{s=1}^{k}\phi^{-1}\Big(\lambda\sum_{l=s}^{T}p(l)\widetilde{g}(u(l)-w(l))-\phi(\Delta w(s-1))\Big)+\theta w(k)\leqslant\\
&\sum_{s=1}^{k}\Big\{\phi^{-1}\Big(\lambda\sum_{l=s}^{T}p(l)\widetilde{g}(u(l)-w(l))\Big)+\phi^{-1}(-\phi(\Delta w(s-1)))+\phi^{-1}(\lambda MT\bar{p})\Big\}+w(k)\leqslant\\
&\sum_{s=1}^{k}\phi^{-1}\Big(\lambda\sum_{l=s}^{T}p(l)\widetilde{g}(u(l)-w(l))\Big)+(T+1)\phi^{-1}(\lambda MT\bar{p})\leqslant\\
&(T+1)\phi^{-1}(\lambda T\bar{p}h(r))+(T+1)\phi^{-1}(\lambda MT\bar{p})\leqslant\\
&\frac{r}{2}+\frac{r}{2}=r
\end{aligned}$$

Thus, we have $r=\|u\|<r$, a contradiction.

Next, we show that there exist constants $R>r$, $\beta>R$ such that all solutions $u\in K$ of $u=Fu+\theta\beta$, $0<\theta<1$, satisfy $\|u\|\neq R$. Such u satisfies the following boundary value problem

$$\begin{cases}L[u-w](k-1)+L[w](k-1)=\\-\lambda p(k)\widetilde{g}(u(k)-w(k)),k\in[1,T]_Z,\\u(0)=\theta\beta=u(T+1)\end{cases} \tag{11}$$

Let $\|u\|=u(k_0)$, $k_0\in[1,T]_Z$, and $\|u\|>r$. We distinguish two cases to show that there exists $R>r$ such that $\|u\|<R$.

Case one: $k_0\geqslant\dfrac{T+1}{2}$. Let v solve the following problem

$$\begin{cases}L[v-w](k-1)+L[w](k-1)=0,\\\qquad k\in[1,k_0-1]_Z,\\v(0)=\theta\beta,v(k_0)=\|u\|\end{cases} \tag{12}$$

Then we have

$$\begin{cases}L[u-w](k-1)-L[v-w](k-1)=\\-\lambda p(k)\widetilde{g}(u(k)-w(k)),k\in[1,k_0-1]_Z,\\(u-v)(0)=0=(u-v)(k_0)\end{cases}$$

It follows by Lemma 2 that $u(k)>v(k)$, $k\in[1,k_0-1]_Z$. Note that the solution v of (12) can be expressed as

$$v(k)=\|u\|-\sum_{l=k+1}^{k_0}\{\phi^{-1}[C_v-\phi(\Delta w(l-1))]+\Delta w(l-1)\},k\in[0,k_0]_Z \tag{13}$$

where C_v satisfies $v(0)=\theta\beta$. Thus,

$$\|u\|=\theta\beta+\sum_{l=1}^{k_0}\{\phi^{-1}[C_v-\phi(\Delta w(l-1))]+\Delta w(l-1)\} \tag{14}$$

We claim that $0\leqslant C_v\leqslant\phi\Big(\dfrac{2\|u\|}{k_0}\Big)$. In fact, if $C_v<0$, then $u(0)=v(0)>\|u\|-\sum\limits_{l=1}^{k_0}\{\phi^{-1}(-\phi(\Delta w(l-1)))+\Delta w(l-1)\}=\|u\|$. It is a contradiction. If

$C_v > \phi\left(\frac{2\|u\|}{k_0}\right)$, then, by (10) and Lemma 3, we have that for $k \in [1, k_0]_Z$,

$$C_v > \phi\left(\frac{r}{T+1}\right) > \lambda TM\bar{p} \geqslant \phi(\Delta w(k-1))$$

It follows by Lemma 3 and 4 that for $k \in [1, k_0]_Z$,

$$\begin{aligned}\phi^{-1}(C_v) &= \phi^{-1}[C_v - \phi(\Delta w(k-1)) + \phi(\Delta w(k-1))] \leqslant \\ &\phi^{-1}(C_v - \phi(\Delta w(k-1))) + \Delta w(k-1) + \phi^{-1}(\lambda MT\bar{p})\end{aligned}$$

Thus, by (10), we have that for $k \in [1, k_0]_Z$,

$$\begin{aligned}&\phi^{-1}(C_v - \phi(\Delta w(k-1))) + \Delta w(k-1) \geqslant \\ &\phi^{-1}(C_v) - \phi^{-1}(\lambda MT\bar{p}) > \frac{2\|u\|}{k_0} - \frac{r}{T+1} > \frac{\|u\|}{k_0}\end{aligned}$$

which yields from (14) that $\|u\| > \theta\beta + \|u\|$, a contradiction.

Let $k_1 = \left\lceil \frac{2}{3}k_0 \right\rceil$ ($\lceil x \rceil$ denotes the minimum of integers not less than x). Since $T > 5$, we know $k_0 - k_1 \geqslant 1$. Then we have from (13), (10) and Lemma 4 that for $k \in [k_1, k_0]_Z$,

$$\begin{aligned}v(k) &\geqslant \|u\| - \sum_{l=k+1}^{k_0}\left\{\phi^{-1}\left[\phi\left(\frac{2\|u\|}{k_0}\right) + \phi(-\Delta w(l-1))\right] + \Delta w(l-1)\right\} \geqslant \\ &\|u\| - \sum_{l=k+1}^{k_0}\left[\frac{2\|u\|}{k_0} - \Delta w(l-1) + \phi^{-1}(\lambda MT\bar{p}) + \Delta w(l-1)\right] \geqslant \\ &\|u\| - (k_0 - k)\frac{2\|u\|}{k_0} - (T+1)\phi^{-1}(\lambda MT\bar{p}) \geqslant \\ &\frac{1}{3}\|u\| - \frac{1}{8}r > \frac{5}{24}\|u\|\end{aligned}$$

It follows that for $k \in [k_1, k_0]_Z$,

$$\begin{aligned}u(k) - w(k) &\geqslant v(k) - w(k) > \\ &\frac{5}{24}\|u\| - w(k) > \frac{1}{12}\|u\|\end{aligned}$$

since for $k \in [0, T+1]_Z$,

$$\begin{aligned}w(k) &= \sum_{l=1}^{k}\Delta w(l-1) \leqslant \\ &(T+1)\phi^{-1}(\lambda MT\bar{p}) < \frac{1}{8}\|u\| \qquad (15)\end{aligned}$$

Note that

$$\begin{aligned}u(k) = &\sum_{s=1}^{k}\phi^{-1}\left(C_u - \lambda\sum_{l=1}^{s-1}p(l)\tilde{g}(u(l) - w(l)) - \right. \\ &\left.\phi(\Delta w(s-1))\right) + w(k) + \theta\beta\end{aligned}$$

and (8) holds with $\tilde{u}$ being replaced by u. Then we have

$$\begin{aligned}\|u\| &\geqslant u(k_1) \geqslant \\ &\sum_{s=1}^{k_1}\phi^{-1}\left(\lambda\left(\sum_{l=1}^{k_0-1} - \sum_{l=1}^{s-1}\right)p(l)\tilde{g}(u(l) - w(l)) - \right. \\ &\left.\phi(\Delta w(s-1))\right) \geqslant \\ &\sum_{s=1}^{k_1}\phi^{-1}\left(\lambda\sum_{l=k_1}^{k_0-1}p(l)\tilde{g}(u(l) - w(l)) - \right. \\ &\left.\phi(\Delta w(s-1))\right) \geqslant \\ &\frac{2}{3}k_0\phi^{-1}\left((k_0 - k_1)\lambda\underline{p}\inf_{x \geqslant \frac{1}{12}\|u\|}\frac{\tilde{g}(x)}{\phi(x)}\phi\left(\frac{\|u\|}{12}\right) - \lambda MT\bar{p}\right) \geqslant \\ &\frac{1}{3}(T+1)\phi^{-1}\left(\lambda\underline{p}\inf_{x \geqslant \frac{1}{12}\|u\|}\frac{\tilde{g}(x)}{\phi(x)}\phi\left(\frac{\|u\|}{12}\right) - \lambda MT\bar{p}\right)\end{aligned}$$

Assume $0 < f^{\infty} < \infty$. Take $\varepsilon > 0$ sufficiently small, such that $\lambda > \frac{2\varphi(12)}{\underline{p}(f^{\infty} - \varepsilon)\psi\left(\frac{T+1}{3}\right)}$. For sufficiently large $\|u\|$, we know by (A2) that $\inf_{x \geqslant \frac{1}{12}\|u\|}\frac{\tilde{g}(x)}{\phi(x)} > f^{\infty} - \varepsilon$, and by the fact $\lim_{u \to \infty}\phi(u) = \infty$ that $\lambda MT\bar{p} < \frac{1}{2}\lambda\underline{p}f^{\infty}\phi\left(\frac{\|u\|}{12}\right)$. Thus, for sufficiently large $\|u\|$,

$$\|u\| > \frac{1}{3}(T+1)\phi^{-1}\left(\frac{1}{2}\lambda\underline{p}(f^{\infty} - \varepsilon)\phi\left(\frac{\|u\|}{12}\right)\right)$$

It follows that for sufficiently large $\|u\|$,

$$\begin{aligned}\phi\left(\frac{3}{T+1}\|u\|\right) &> \frac{1}{2}\lambda\underline{p}(f^{\infty} - \varepsilon) \\ \phi\left(\frac{\|u\|}{12}\right) &> \frac{\varphi(12)}{\psi\left(\frac{T+1}{3}\right)}\phi\left(\frac{\|u\|}{12}\right) \qquad (16)\end{aligned}$$

By (2), we know

$$\phi(\|u\|) \geqslant \phi\left(\frac{3}{T+1}\|u\|\right)\psi\left(\frac{3}{T+1}\right)$$

and

$$\phi(\|u\|) \leqslant \phi\left(\frac{1}{12}\|u\|\right)\varphi(12)$$

Then, (16) implies $\phi(\|u\|) > \phi(\|u\|)$, a contradiction. If $f^{\infty} = \infty$, then we can choose $N > \frac{2\varphi(12)}{\lambda\underline{p}\psi\left(\frac{T+1}{3}\right)}$ $(0 < \lambda < \lambda_r)$ such that for sufficiently large $\|u\|$,

$$\|u\| > \frac{1}{3}(T+1)\phi^{-1}\left(\frac{1}{2}\lambda\underline{p}N\phi\left(\frac{\|u\|}{12}\right)\right)$$

Thus we again have

$$\phi\left(\frac{3}{T+1}\|u\|\right) > \frac{\varphi(12)}{\psi\left(\frac{T+1}{3}\right)}\phi\left(\frac{\|u\|}{12}\right)$$

Therefore, there exists $R_1 > r$, independent of u, θ, and β such that $\|u\| < R_1$.

Case two: $k_0 < \frac{T+1}{2}$. Let $\tilde{v}$ be the solution of

$$\begin{cases} L[\tilde{v}-w](k-1)+L[w](k-1)=0, \\ \qquad k\in[k_0+1,T]_Z, \\ \tilde{v}(k_0)=\|u\|, \tilde{v}(T+1)=\theta\beta \end{cases} \tag{17}$$

Then we have by Lemma 2 that $u(k)>\tilde{v}(k), k\in[k_0+1,T]_Z$. Note that

$$\tilde{v}(k)=\|u\|-\sum_{l=k_0+1}^{k}\{\phi^{-1}[C_{\tilde{v}}+\phi(\Delta w(l-1))]-\Delta w(l-1)\}, k\in[k_0,T+1]_Z \tag{18}$$

where $C_{\tilde{v}}$ satisfies $\tilde{v}(T+1)=\theta\beta$. Thus,

$$\|u\|=\theta\beta+\sum_{l=k_0+1}^{T+1}\{\phi^{-1}[C_{\tilde{v}}+\phi(\Delta w(l-1))]-\Delta w(l-1)\} \tag{19}$$

Similar to the arguments in Case one, it is easy to check that $0\leqslant C_{\tilde{v}}\leqslant\phi\left(\frac{2\|u\|}{T+1-k_0}\right)$.

Let $k_2=\left\lfloor\frac{T+1+2k_0}{3}\right\rfloor$ ($\lfloor x\rfloor$ denotes the integer part of x). It is easy to see that $k_2-k_0\geqslant 1$ by $T>5$. Hence, we have from (18), (10) and Lemma 4 that for $k\in[k_0,k_2]_Z$,

$$\tilde{v}(k)\geqslant\|u\|-\sum_{l=k_0+1}^{k}\left\{\phi^{-1}\left[\phi\left(\frac{2\|u\|}{T+1-k_0}\right)+\phi(\Delta w(l-1))\right]-\Delta w(l-1)\right\}\geqslant$$

$$\|u\|-\sum_{l=k_0+1}^{k}\left[\frac{2\|u\|}{T+1-k_0}+\Delta w(l-1)+\phi^{-1}(\lambda MT\bar{p})-\Delta w(l-1)\right]\geqslant$$

$$\|u\|-(k-k_0)\frac{2\|u\|}{T+1-k_0}-(T+1)\phi^{-1}(\lambda MT\bar{p})>\frac{1}{3}\|u\|-\frac{1}{8}\|u\|=\frac{5}{24}\|u\|$$

It follows by (15) that for $k\in[k_0,k_2]_Z, u(k)-w(k)>\frac{1}{12}\|u\|$.

Note that

$$u(k)=\sum_{s=k}^{T}\phi^{-1}\left(\tilde{C}_u-\lambda\sum_{l=s+1}^{T}p(l)\tilde{g}(u(l)-w(l))+\phi(\Delta w(s))\right)+w(k)+\theta\beta, k\in[0,T+1]_Z$$

and (9) still holds with $\tilde{u}$ being replaced by u. Then we have that

$$\|u\|\geqslant u(k_2)\geqslant$$

$$\sum_{s=k_2}^{T}\phi^{-1}\left(\lambda\sum_{l=k_0+1}^{s}p(l)\tilde{g}(u(l)-w(l))+\phi(\Delta w(s))\right)\geqslant$$

$$\sum_{s=k_2}^{T}\phi^{-1}\left(\lambda\sum_{l=k_0+1}^{k_2}p(l)\tilde{g}(u(l)-w(l))+\phi(\Delta w(s))\right)\geqslant(T+1-k_2)\phi^{-1}\left((k_2-k_1)\cdot\lambda\underline{p}\inf_{x\geqslant\frac{1}{12}\|u\|}\frac{\tilde{g}(x)}{\phi(x)}\phi\left(\frac{\|u\|}{12}\right)-\lambda MT\bar{p}\right)\geqslant$$

$$\frac{1}{3}(T+1)\cdot\phi^{-1}\left(\lambda\underline{p}\inf_{x\geqslant\frac{1}{12}\|u\|}\frac{\tilde{g}(x)}{\phi(x)}\phi\left(\frac{\|u\|}{12}\right)-\lambda MT\bar{p}\right)$$

Using similar arguments as before, there exists $R_2>r$, independent of u,θ and β such that $\|u\|<R_2$.

In summary, by Lemma 1, for $0<\lambda<\lambda_r$, F has a fixed point $\tilde{u}$ in K with $\|\tilde{u}\|\geqslant r$, which is a positive solution of (5).

Step II. We shall prove that $\tilde{u}>w$ on $[1,T]_Z$.

Let $\|\tilde{u}\|=\tilde{u}(k_0), k_0\in[1,T]_Z$. We distinguish two cases that (i) $1\leqslant k\leqslant k_0$ and (ii) $k_0\leqslant k\leqslant T$ to finish the proof.

(i) If $1\leqslant k\leqslant k_0$, we use the expression (6), i. e.,

$$\tilde{u}(k)=\sum_{s=1}^{k}\phi^{-1}\left(C_{\tilde{u}}-\lambda\sum_{l=1}^{s-1}p(l)\tilde{g}(\tilde{u}(l)-w(l))-\phi(\Delta w(s-1))\right)+w(k), k\in[0,T+1]_Z$$

where $C_{\tilde{u}}$ satisfies $\tilde{u}(T+1)=0$. By (8), there exists $0\leqslant\varepsilon\leqslant 1$ such that

$$\begin{aligned} C_{\tilde{u}}&=(1-\varepsilon)\lambda\sum_{l=1}^{k_0-1}p(l)\tilde{g}(\tilde{u}(l)-w(l))+\varepsilon\lambda\sum_{l=1}^{k_0}p(l)\tilde{g}(\tilde{u}(l)-w(l))= \\ &\lambda\sum_{l=1}^{k_0-1}p(l)\tilde{g}(\tilde{u}(l)-w(l))+\varepsilon\lambda p(k_0)\tilde{g}(\tilde{u}(k_0)-w(k_0)) \end{aligned} \tag{20}$$

For $k\in[0,k_0]_Z$, let

$$G(k)=\sum_{s=1}^{k}\phi^{-1}\left(\lambda\sum_{l=s}^{k_0-1}p(l)\tilde{g}(\tilde{u}(l)-w(l))+\varepsilon\lambda p(k_0)\tilde{g}(\tilde{u}(k_0)-w(k_0))\right)$$

It is easy to see that

$$\Delta G(k-1)=\phi^{-1}\left(\lambda\sum_{l=k}^{k_0-1}p(l)\tilde{g}(\tilde{u}(l)-w(l))+\varepsilon\lambda p(k_0)\tilde{g}(\tilde{u}(k_0)-w(k_0))\right)$$

is nonincreasing on $[1,k_0]_Z$. That is to say that G is a discrete concave function on $[0,k_0]$. Then we have

$$G(k)\geqslant\frac{k}{k_0}G(k_0), k\in[0,k_0]_Z \tag{21}$$

From (20) and Lemma 4 we have that

$$r \leqslant \tilde u(k_0) = \sum_{s=1}^{k_0}\phi^{-1}\Big(\lambda\sum_{l=s}^{k_0-1}p(l)\tilde g(\tilde u(l)-w(l)) + \varepsilon\lambda p(k_0)\tilde g(\tilde u(k_0)-w(k_0))\Big) - \phi(\Delta w(s-1))) + w(k_0) \leqslant \sum_{s=1}^{k_0}\Big\{\phi^{-1}\Big(\lambda\sum_{l=s}^{k_0-1}p(l)\tilde g(\tilde u(l)-w(l)) + \varepsilon\lambda p(k_0)\tilde g(\tilde u(k_0)-w(k_0))\Big) - \Delta w(s-1) + \phi^{-1}(\lambda MT\bar p)\Big\} + w(k_0) < G(k_0) + (T+1)\phi^{-1}(\lambda TM\bar p)$$

It follows by (10) that

$$G(k_0) > r - (T+1)\phi^{-1}(\lambda TM\bar p) \geqslant \frac{7}{8}r \tag{22}$$

and hence by (21) that $G(k) > \frac{7k}{8(T+1)}r$. Therefore, from (20), Lemma 3 and the second inequality of Lemma 4, we have

$$\tilde u(k) - w(k) = \sum_{s=1}^{k}\phi^{-1}\Big(C_u - \lambda\sum_{l=1}^{s-1}p(l)\tilde g\cdot(\tilde u(l)-w(l)) - \phi(\Delta w(s-1))\Big) = \sum_{s=1}^{k}\phi^{-1}\Big(\lambda\sum_{l=s}^{k_0-1}p(l)\tilde g(\tilde u(l)-w(l)) + \varepsilon\lambda p(k_0)\tilde g(\tilde u(k_0)-w(k_0)) - \phi(\Delta w(s-1))\Big) \geqslant \sum_{s=1}^{k}\phi^{-1}\Big(\lambda\sum_{l=s}^{k_0-1}p(l)\tilde g(\tilde u(l)-w(l)) + \varepsilon\lambda p(k_0)\tilde g(\tilde u(k_0)-w(k_0)) - \lambda MT\bar p\Big) \geqslant G(k) - 2k\phi^{-1}(\lambda MT\bar p) > \frac{7k}{8(T+1)}r - \frac{2k}{8(T+1)}r = \frac{5k}{8(T+1)}r > 0, \quad k \in [1,k_0]_Z$$

(ii) The second case $k_0 \leqslant k \leqslant T$ can be similarly dealt with. We use the expression (7), i. e.,

$$\tilde u(k) = \sum_{s=k}^{T}\phi^{-1}\Big(\tilde C_{\tilde u} - \lambda\sum_{l=s+1}^{T}p(l)\cdot\tilde g(\tilde u(l)-w(l)) + \phi(\Delta w(s))\Big) + w(k), \quad k \in [0,T+1]_Z$$

where $\tilde C_{\tilde u}$ satisfies $\tilde u(0) = 0$. By (9), there exists $0 \leqslant \varepsilon \leqslant 1$ such that

$$C_{\tilde u} = \lambda\sum_{l=k_0+1}^{T}p(l)\tilde g(\tilde u(l)-w(l)) + \varepsilon\lambda p(k_0)\tilde g(\tilde u(k_0)-w(k_0)) \tag{23}$$

For $k \in [k_0,T+1]_Z$, let

$$\tilde G(k) = \sum_{s=k}^{T}\phi^{-1}\Big(\lambda\sum_{l=k_0+1}^{s}p(l)\tilde g(\tilde u(l)-w(l)) + \varepsilon\lambda p(k_0)\tilde g(\tilde u(k_0)-w(k_0))\Big)$$

Then one can find that $\Delta\tilde G(k-1)$ is nonincreasing on $[k_0+1,T+1]_Z$, which implies that

$$\frac{\tilde G(k)-\tilde G(k_0)}{k-k_0} \geqslant \frac{\tilde G(T+1)-\tilde G(k_0)}{T+1-k_0}, \quad k \in [k_0,T+1]_Z$$

That is to say

$$\tilde G(k) \geqslant \frac{T+1-k}{T+1-k_0}\tilde G(k_0), k \in [k_0,T]_Z \tag{24}$$

Thus we have by (23) and Lemma 4 that

$$r \leqslant \tilde u(k_0) < \tilde G(k_0) + (T+1-k_0)\phi^{-1}(\lambda TM\bar p)$$

It follows by (10) that

$$\tilde G(k_0) > r - \frac{T+1-k_0}{8(T+1)}r$$

which implies by (24) that $\tilde G(k) > \frac{7(T+1-k)}{8(T+1)}r, k \in [k_0,T]_Z$. Therefore, by (23), Lemma 3 and the second inequality of Lemma 4 that

$$\tilde u(k) - w(k) \geqslant \tilde G(k) - 2(T+1-k)\phi^{-1}(\lambda MT\bar p) > \frac{5(T+1-k)}{8(T+1)}r > 0, k \in [k_0,T]_Z$$

In summary, we have in each case that $\tilde u(k) - w(k) > 0$ on $[1,T]_Z$. The proof is complete.

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